Discrete Random Variable and Its Distributions

1 Definition of a Discrete Random Variable

A **discrete random variable** X is one that takes a countable number of distinct values. It is described using a probability mass function (PMF), which gives the probability of each possible value.

The probability that X takes a specific value x_i is given by:

$$P(X = x_i) = p_X(x_i), \tag{1}$$

where $p_X(x)$ satisfies the following properties: 1. $0 \leq p_X(x) \leq 1$ for all x. 2. The sum of all probabilities equals 1:

$$\sum_i p_X(x_i) = 1. \tag{2}$$

2 Cumulative Distribution Function (CDF)

The cumulative distribution function (CDF) of a discrete random variable is defined as:

$$F_X(x) = P(X \leq x) = \sum_{x_i \leq x} p_X(x_i). \tag{3}$$

3 Expectation and Variance

3.1 Expectation (Mean)

$$E[X] = \sum_i x_i p_X(x_i). \tag{4}$$

3.2 Variance

$$Var(X) = E[X^2] - (E[X])^2, \tag{5}$$

where

$$E[X^2] = \sum_i x_i^2 p_X(x_i). \tag{6}$$

4 Common Discrete Distributions

4.1 Bernoulli Distribution $Bern(p)$

$$P(X = x) = \begin{cases} p, & x = 1, \\ 1 - p, & x = 0. \end{cases} \tag{7}$$

Mean: $E[X] = p$, Variance: $Var(X) = p(1 - p)$.

4.2 Binomial Distribution $Bin(n, p)$

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}, \quad k = 0, 1, \ldots, n. \tag{8}$$

Mean: $E[X] = np$, Variance: $Var(X) = np(1 - p)$.

4.3 Poisson Distribution $Pois(\lambda)$

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad k = 0, 1, 2, \ldots. \tag{9}$$

Mean: $E[X] = \lambda$, Variance: $Var(X) = \lambda$.

4.4 Geometric Distribution $Geom(p)$

$$P(X = k) = (1 - p)^{k-1} p, \quad k = 1, 2, 3, \ldots. \tag{10}$$

Mean: $E[X] = \frac{1}{p}$, Variance: $Var(X) = \frac{1-p}{p^2}$.

4.5 Hypergeometric Distribution $Hypergeo(N, K, n)$

$$P(X = k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}, \quad \max(0, n - (N - K)) \leq k \leq \min(n, K). \tag{11}$$

Mean: $E[X] = n\frac{K}{N}$, Variance: $Var(X) = n\frac{K}{N}\frac{N-K}{N}\frac{N-n}{N-1}$.

5 Law of Large Numbers and Central Limit Theorem

5.1 Law of Large Numbers (LLN)

The Law of Large Numbers states that as the number of observations increases, the sample mean of a sequence of independent and identically distributed (i.i.d.)

random variables converges to the expected value. Formally, if $X_1, X_2, \ldots, X_n$ are i.i.d. with expectation $E[X]$, then:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X_i = E[X] \quad \text{(almost surely).} \tag{12}$$

This theorem justifies the idea that larger samples provide better estimates of the true mean.

5.2 Central Limit Theorem (CLT)

The Central Limit Theorem states that the sum (or mean) of a large number of i.i.d. random variables, regardless of their original distribution, approaches a normal distribution as the sample size grows. Specifically, if $X_1, X_2, \ldots, X_n$ are i.i.d. with mean μ and variance σ^2, then:

$$\frac{\sum_{i=1}^{n} X_i - n\mu}{\sigma\sqrt{n}} \xrightarrow{d} N(0,1) \quad \text{as } n \to \infty. \tag{13}$$

This theorem is fundamental in statistics, as it allows normal approximations for many types of problems involving sums of random variables.

6 Conclusion

Discrete random variables are fundamental in probability and statistics. The key aspects include the use of probability mass functions, cumulative distribution functions, and properties like expectation and variance. Various distributions such as Bernoulli, Binomial, Poisson, Geometric, and Hypergeometric model different real-world scenarios. The Law of Large Numbers ensures that sample averages converge to the true mean, while the Central Limit Theorem explains why normal distribution approximations are widely applicable in statistical inference.